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# Restricted Range Approximation with Side Conditions

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Let $X$ be a compact subset of $[a, b]$, and let $C(X)$ denote the Banach space of all real-valued continuous functions defined on $X$. Let $\Pi$ denote the set of polynomials in $C(X)$. Consider two extended real-valued functions $\ell$ and $u$ defined on $X$ which satisfy the following conditions.

(i) $\ell$ may take on the value $-\infty$, but never $+\infty$;

(ii) $u$ may take on the value $+\infty$, but never $-\infty$;

(iii) there exist $\underline{\ell}, \underline{u}$ continuous on $[a, b]$ such that $\ell(x) \leqslant \underline{\ell}(x) \leqslant \underline{u}(x) \leqslant u(x)$ for all $x \in X$.

(iv) the $\underline{\ell}, \underline{u}$ of (iii) may be chosen so that $\underline{\ell}(x) = \underline{u}(x)$ at a finite number of points of $[a, b]$ only; and moreover,

(v) if $\underline{\ell}(y) = \underline{u}(y)$, then there exist constants $\xi, \xi', \eta, \psi$ (with $\eta > 0$, $\xi \neq \xi'$) and a positive integer $\alpha$ such that, for $x \in N_\eta(y)$,

$$\begin{aligned} R(\psi, \underline{\ell}(x) - \underline{\ell}(y)) \leqslant \xi'(x-y)^\alpha \leqslant \xi(x-y)^\alpha \\ \leqslant R(\psi, \underline{u}(x) - \underline{u}(y)), \end{aligned} \tag{1}$$

where $R(\psi, \cdot)$ rotates the $(x, u)$-plane by an angle $\psi$ at the point $(y, \underline{\ell}(y) = \underline{u}(y))$.

Let $\Pi^* = \Pi^*(\ell, u) = \{p \in \Pi : l \leqslant p \leqslant u \text{ on } X\}$. We may now state the restricted range approximation scheme as follows.

RESTRICTED RANGE APPROXIMATION SCHEME. Given $f \in C(X)$, approximate $f$ by polynomials $p \in \Pi^*$.

This approximation scheme has been considered by several authors (e.g., [1, 4–8]). Between them the questions of existence, uniqueness, characterization, and nontriviality of best restricted range (polynomial) approxima-

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tions have been considered, and some algorithms given. In this paper we consider the related

*Restricted Range Approximation with Side Conditions Scheme. Given $f \in C(X)$ and bounded linear functionals $x_1^*,..., x_n^*$, approximate $f$ by polynomials $p \in \Pi^*$ for which $x_i^*p = x_i^*f$ $(i = 1,..., n)$.*

We characterize those $n$-tuples of linear functionals for which one may approximate any continuous function $f$ arbitrary closely in the restricted range approximation with side conditions (RRAS) scheme, for any permissible pair of bounding functions $\ell$, $u$. For simplicity we will assume that $X = [a, b]$ below.

## 1. Preliminaries

In [1], conditions (i)–(v) are shown to be necessary and sufficient in order that the restricted range approximation (RRA) scheme is not trivial. Calling pairs of bounding functions $\ell$, $u$ satisfying conditions (i)–(v) *permissible*, we thus have

PROPOSITION 1.1 [1]. *Suppose $f \in C[a, b]$ and $\ell, u$, permissible bounding functions, are such that $\ell \leqslant f \leqslant u$. Then given $\epsilon > 0$, there exists a $p_\epsilon \in \Pi^*(\ell, u)$ such that $\| f - p_\epsilon \| < \epsilon$.*

DEFINITION 1.1 [2]. Suppose $x_1^*,..., x_n^*$ is a set of bounded linear functionals for which no nontrivial linear combination $\sum_{i=1}^{n} a_i x_i^*$ is ever a positive linear functional on $C[a, b]$. Such sequences $x_1^*,..., x_n^*$ are said to be *span indefinite*.

PROPOSITION 1.2 [2]. *Suppose $x_1^*,..., x_n^*$ are span indefinite on $C[a, b]$. Then there exists a polynomial $p \in \Pi$ for whixh* (i) *$p(x) \geqslant 1$ on $[a, b]$ and* (ii) $x_j^*p = 0$ $(j = 1,..., n)$.

*Remark* 1.1. (a) Any bounded linear functional $x^*$ on $C[a, b]$ has a unique decomposition into the difference of two positive linear functionals (called the positive and negative parts of $x^*$);

$$x^* = x^{+*} - x^{-*}, \qquad \| x^* \| = \| x^{+*} \| + \| x^{-*} \| .$$

(b) A functional $x^*$ is purely atomic in case the associated Borel measure [9, p. 34] is purely atomic.

(c) A functional $y^*$ is perfect nowhere dense in case (i) supp $y^*$ is the countable union of perfect, nowhere dense subsets of $[a, b]$ having positive Lebesgue measure, and (ii) $y^*$ has no atoms.

(d) A functional $z^*$ is of purely continuum type in case (i) $z^*$ has no atoms, and (ii) $\| z^* \circ \chi_J \| = 0$ for every perfect, nowhere dense subset $J$ of $[a, b]$.

(e) Any bounded linear functional $x^*$ has a unique decomposition into the sum of a purely atomic, a perfect nowhere dense, and a purely continuum linear functional;

$$x^* = w^* + y^* + z^*, \; \| x^* \| = \| w^* \| + \| y^* \| + \| z^* \|.$$

(f) If $w^*$ is purely atomic and $t \in \operatorname{supp} w^*$, then $t$ can have a zero weight only if $t$ is a cluster point of (a countably infinite number of) atoms $t_i$ of $w^*$ having nonzero weights.

By the *nodes* of a pair of permissible bounding functions $\ell, u$ we mean the (finitely many) points $t$ of $[a, b]$ for which $\ell(t) = u(t)$. We use $\operatorname{card}(T) \geqslant \aleph_0$ to mean $T$ has infinitely many points, and $N_\delta(T)$ for a $\delta$-neighborhood of $T$.

## 2. RRAS Functionals

Definition 2.1. Suppose $x^*$ is a bounded linear functional on $C[a, b]$ $x^*$ is said to be a *RRAS functional* in case given $\epsilon > 0, f \in C[a, b]$ and permissible $\ell, u$ for which $\ell \leqslant f \leqslant u$ on $[a, b]$ there necessarily exists a polynomial $p \in \Pi$ for which (i) $x^*p = x^*f$, (ii) $\ell \leqslant p \leqslant u$ on $[a, b]$, and (iii) $\| f - p \| < \epsilon$.

Theorem 2.1. *A bounded linear functional $x^*$ on $C[a, b]$ is a* RRAS *functional if and only if*

$$\operatorname{card}(\operatorname{supp} x^{+*} \cap \operatorname{supp} x^{-*}) \geqslant \aleph_0 . \tag{1}$$

*Proof.* Set $A = \operatorname{supp} x^{+*}$, $B = \operatorname{supp} x^{-*}$. If $A$ and $B$ are disjoint, consider $\ell(x) = -1 = -u(x)$ and any $f \in C[a, b]$ for which $f(x)$ is one on $\operatorname{supp} x^{+*}$ and minus one on $\operatorname{supp} x^{-*}$. Since $f$ is extremal for $x^*$ from $C[a, b]$, any $p \in \Pi$, $\| p \| \leqslant 1$ for which $x^*p = x^*f$ must be one on $\operatorname{supp} x^{+*}$ and minus one on $\operatorname{supp} x^{-*}$. Since a nonconstant polynomial can attain its norm at most finitely often, either $A$ and $B$ both have finite cardinality or else one of $A$ and $B$ is empty. Suppose that $B = \varnothing$ but $A$ is not finite. Let $t \in A$ be a cluster point of $A$ and consider $u(x) = f(x) = - | x - t |, \ell(x) \equiv -\infty$. Since $x^*$ is a positive linear functional, any $p \in \Pi$ for which $\ell \leqslant p \leqslant u$ and $x^*p = x^*f$ will have to equal $f$ on $A$, and hence at $t$. But no such $p \in \Pi$ can exist. Suppose that $A$ and $B$ are both nonempty finite point sets, $x^* = \sum_{i=1}^{m} \alpha_i e_{s_i}$ for some nonzero constants $\alpha_i$ and distinct points $s_i \in [a, b]$. Without loss of generality suppose $\alpha_1 < 0$ and consider $\ell(x) = | x - s_1 | = u(x) - 1$. Let $f$ be any continuous function on $[a, b]$ for which $f(s_i) = \ell(s_i)$ if $\alpha_i < 0$, $u(s_i)$ if $\alpha_i > 0$. Again any polynomial $p \in \Pi$ for which $\ell \leqslant p \leqslant u$ and $x^*p = x^*f$ must interpolate $f$ at the $s_i$. But no polynomial can simultaneously interpolate $f$ at $x_1$ and be inside the bounding functions $\ell, u$.

Hence suppose $A \cap B = \{t_1, ..., t_u\}$ is a nonempty finite point set. By the definition of the positive and negative parts of a linear functional at least one of $A$ and $B$ has to be infinite. Consider bounding functions $\ell(x)$, $u(x)$ which (i) are equal at each $t_i$, $\ell(t_i) = u(t_i) = 0$, (ii) in some $\delta$-neighborhood of each $t_i$, $\ell(x)$ coincides with the function $- | x - t_i |$, and $u(x)$ coincides with the function $2 | x - t_i |$, (iii) are not equal if not at a $t_i$, $\ell(x) < 0 < u(x)$ if $x \notin A \cap B$, and (iv) are continuous on $[a, b]$. Choose a nonpolynomial (if possible) $f \in C[a, b]$ for which (i) $f(t_i) = 0$ $(i = 1, ..., n)$, (ii) $f(x) = \ell(x)$ if $x \in B$, and (iii) $f(x) = u(x)$ if $x \in A$. By construction $f$ is extremal for $x^*$ from those continuous functions $g \in C[a, b]$ which lie within the bounding functions $\ell, u$. Hence any $p \in \Pi$ for which $\ell \leqslant p \leqslant u$ and $x^*p = x^*f$ must (without loss of generality) equal $\ell(x)$ on $A$ and $u(x)$ on $B$. Since $A$ or $B$ is infinite, any such polynomial is unique. Hence $F$ can be approximated arbitrarily closely by such polynomials if and only if $f$ is already a polynomial, in which case one of $A$ and $B$ must be a singleton (say $A$) and the other infinite. Now consider $\ell(t_1) = 0$, $u(x) = f(x)$ any nonpolynomial for which $u(t_1) = 0$ and $\ell$, $u$ are permissible. Any $p \in \Pi$ for which $\ell \leqslant p \leqslant u$ and $x^*p = x^*f$ is again uniquely determined, but this time $f$ is not a polynomial.

Conversely, suppose $A \cap B$ is infinite.

LEMMA 2.1. *Suppose $F \in C[a, b]$. Suppose $L$, $U$ are permissible bounding functions for which $L \leqslant F \leqslant U$. Then there exist $G, H \in C[a, b]$ such that $L \leqslant G, H \leqslant U$ and $x^*G < x^*F < x^*H$.*

Suppose $f \in C[a, b]$ and permissible $\ell$, $u$ such that $\ell \leqslant f \leqslant u$ are fixed. For $\epsilon > 0$ arbitrary, let

$$L_\epsilon(x) = f(x) - \epsilon, \text{ if } (f - \ell)(x) > \epsilon \quad U_\epsilon(x) = f(x) + \epsilon, \text{ if } (u - f)(x) > \epsilon$$
$$= \ell(x), \quad \text{otherwise:} \qquad\qquad = u(x), \quad \text{otherwise.}$$

At each node of $L_\epsilon$, $U_\epsilon$, we have $L_\epsilon(x) = \ell(x)$, $U_\epsilon(x) = u(x)$, so the pair $L_\epsilon$, $U_\epsilon$ is permissible. By Lemma 2.1 there are $G_\epsilon$, $H_\epsilon \in C[a, b]$ for which $L_\epsilon \leqslant G_\epsilon$, $H_\epsilon \leqslant U_\epsilon$, and $x^*G_\epsilon < x^*f < x^*H_\epsilon$. Let $\eta = \min\{x^*(H_\epsilon - f), x^*(f - G_\epsilon)\}$. By Proposition 1.1 there are polynomials $p_\epsilon$, $q_\epsilon$ for which $L_\epsilon \leqslant p_\epsilon, q_\epsilon \leqslant U_\epsilon$, $\| G_\epsilon - p_\epsilon \| < \eta \| x^* \|^{-1}/2$, and $\| H_\epsilon - q_\epsilon \| < \eta \| x^* \|^{-1}/2$. Then $x^*p_\epsilon < x^*f < x^*q_\epsilon$, and choose $0 < \lambda < 1$ so that $x^*(\lambda p_\epsilon + (1 -$

$\lambda) q_\epsilon) = x^*f$. Since $\ell \leqslant L_\epsilon \leqslant \lambda p_\epsilon + (1-\lambda) q_\epsilon \leqslant U_\epsilon \leqslant u$, also $\| f - (\lambda p_\epsilon + (1-\lambda) q_\epsilon) \| < \epsilon$ and the proof is complete. ▮

*Proof* (*of Lemma* 2.1). Since $L$, $U$ are permissible, $T = \{x \in [a, b]: L(x) = U(x)\}$ contains at most a finite number of points. Since $A \cap B$ is infinite, $C = (A \cap B)\backslash T$ is then also infinite. Consider the decomposition of Remark 1.1 for $x^*$;

$$\begin{aligned} x^{+*} &= w^{+*} + y^{+*} + z^{+*}, \\ x^{-*} &= w^{-*} + y^{-*} + z^{-*}. \end{aligned} \tag{2}$$

*Case* I. Suppose $t \in C \cap \operatorname{supp} w^{-*}$. If $t$ should be an isolated point of $\operatorname{supp} x^{-*}$, then not only does the atom $e_t$ have some positive weight $\alpha$ in $w^{-*}$ but there even exists an $\epsilon > 0$ for which $x_\epsilon^{-*} = x^{-*} \circ \chi_{(t-\epsilon, t+\epsilon)} = \alpha e_t$. Since $t \notin \operatorname{supp} w^{+*}$, $\| x_\epsilon^{+*} \| = \| x^{+*} \circ \chi_{(t-\epsilon,t+\epsilon)} \| \to 0$ as $\epsilon \to 0^+$. Hence there is an $\eta > 0$ for which $\| x_\epsilon^{+*} \| < \| x_\epsilon^{-*} \| = \alpha$ whenever $0 < \epsilon < \eta$.

Since $t \in C$, let $0 < \psi < \eta$ be such that $\ell(x) < u(x)$ whenever $x \in (t - \psi, t + \psi)$. For $0 < \epsilon < \psi$ choose $g_\epsilon$, $h_\epsilon \in C[a, b]$ so that

(i) $g_\epsilon(t) = f(t) + (u(t) - f(t))/2$,

(ii) $g_\epsilon(x) = f(x)$ if $x \in [a, b]\backslash N_\eta(t)$, and

(iii) $f(x) \leqslant g_\epsilon(x) \leqslant u(x)$ otherwise, while

(iv) $h_\epsilon(t) = f(t) - (f(t) - \ell(t))/2$,

(v) $h_\epsilon(x) = f(x)$ if $x \in [a, b]\backslash N_\eta(t)$, and

(vi) $\ell(x) \leqslant h_\epsilon(x) \leqslant u(x)$ otherwise.

As $\epsilon \to 0^+$, $x^*g \to x^*f - (u - f)(t)/2$, $x^*h_\epsilon \to x^*f + (f - \ell)(t)/2$, and $x^*g_\epsilon$, $x^*h_\epsilon$ are continuous functions of epsilon. If $u(t) > f(t)$, upon choosing $\epsilon > 0$ sufficiently small the desired $G$ of Lemma 2.1 has been found (similarly for $H$ if $f(t) > \ell(t)$). Since $t \in T$ at least one of the above two cases hold: suppose $f(t) < u(t)$ but that $f(t) = \ell(t)$. Considering $0 < \tau < \epsilon < \psi$, choose $h_{\epsilon,\tau} \in C[a, b]$ so that (i) $h_{\epsilon,\tau}(x) = f(x)$ if $x \in N_\tau(t) \cup ([a, b]\backslash N_\epsilon(t))$, (ii) $h_{\epsilon,\tau}(x) = f(x) + (u - f)(x)/2$ if $x = t + (\psi + \eta)/2$, and (iii) $\ell(x) \leqslant h_{\epsilon,\tau}(x) \leqslant f(x)$ otherwise.

Since $t \in A \cap B$, $x^{+*} \circ \chi_{(t-\epsilon,t-\tau) \cup (t+\tau,t+\epsilon)}$ is not the zero functional (for $0 < \tau < \epsilon$ sufficiently small). In particular we can fix $0 < \tau < \epsilon < \psi$ sufficiently small that $x^{+*}h_{\epsilon,\tau} > 0$. But then $x^{-*}h_{\epsilon,\tau} = 0$ implies $x^*h_{\epsilon,\tau} > x^*f$ and $h_{\epsilon,\tau}$ is our desired $H$.

If $t$ is not an isolated point of $\operatorname{supp} x^{-*}$ but still has a positive weight $\alpha$ in $w^{-*}$ a similar construction can be made. For $t \in \operatorname{supp} y^{-*}\backslash\operatorname{supp} z^{-*}$ set $D = [a, b]\backslash\operatorname{supp} y^{-*}$. Since $\operatorname{supp} y^{-*}$ is the union of countably many perfect nowhere dense subsets of $[a, b]$, $D$ is dense in $[a, b]$. If $\operatorname{supp} y^{-*}$ is actually a finite union of perfect nowhere dense subsets of $[a, b]$, then $D$ is also open and we can obtain $G$, $H$ by modifying $f$ on $N_\epsilon(t) \cap E$ and $(N_\epsilon(t)\backslash N_\tau(t)) \cap E$ for some closed subset $E$ of $E$. If supp $y^{-*}$ is not a finite union of perfect nowhere dense subsets of $[a, b]$, suppose $y^{-*} = \sum_{i=1}^{\infty} \beta_i \int_{\Gamma_i} \cdot \, d\mu_i$, where $\Gamma_i$ is perfect nowhere dense of positive Lebesgue measure, $\beta_i > 0$, and $\mu_i$ has total mass one. Since $\sum_{i=1}^{\infty} \beta_i = \| y^{-*} \| < \infty$, $\sum_{i=\nu}^{\infty} \beta_i \to 0$ as $\nu \to \infty$, and hence $\| \sum_{i=\nu}^{\infty} \beta_i \int_{\Gamma_i} \cdot \, d\mu_i \| \to 0$ as $\nu \to \infty$, it is possible to ignore all but finitely many terms of $y^{-*}$ with a negligible change in $x^{-*}$. Hence the above construction can again be carried out.

For $t \in \operatorname{supp} z^{-*}\backslash \operatorname{supp} z^{+*}$, for $\epsilon > 0$ sufficiently small $z^{+*}$ and $w^{+*}$ are the zero functional and $x^{+*}$ reduces to $y^{+*}$. Let $\{D_\xi\}_{\epsilon>0}$ be a decreasing sequence of open subsets of $[a, b]\backslash\{t\}$ whose limit (intersection) is a proper subset of supp $y^{+*}$ having positive measure and not containing $t$. In particular, then, $\| x^{-*} \circ \chi_{D_\xi} \| \to 0$ as $\epsilon \to 0^+$ but $\| x^{+*} \circ \chi_{D_\xi} \| \to \beta > 0$ for some positive constant $\beta$. Letting $E_\xi$ be nonempty closed subsets of $D_\xi$ having positive measure, we can choose $\xi > 0$ sufficiently small so that $z^{-*}$ makes a negligible contribution to $x^* \circ \chi_{E_\xi}$, and the analogous construction of $G$, $H$ will work.

If $t \in \operatorname{supp} w^{-*}$ does not have a positive weight, then being a limit of atoms of $w^{-*}$ having positive weight, choose an atom $t'$ of $w^{-*}$ having positive weight which will also lie in $C$.

*Case* II. $t \in \operatorname{supp} z^{+*}\backslash \operatorname{supp} w^*$. If $t \in \operatorname{supp} z^{-*}$ also, then (locally at $t$) supp $z_\epsilon^{+*} = [t - \epsilon, t]$ and supp $z_\epsilon^{-*} = [t, t + \epsilon]$ or vice versa (provided $\epsilon > 0$ is sufficiently small). To construct $G$, increase $f$ on $[t, t + \epsilon]$ only: for $H$ increase $f$ on $[t - \epsilon, t]$ only). If $y_\epsilon^* = w_\epsilon^* = 0$ would be done. But $t \notin \operatorname{supp} w^*$ means $w_\epsilon^* = 0$ if $\epsilon$ is sufficiently small, and if $y_\epsilon^* \neq 0$ for all $\epsilon > 0$, let $D = [a, b]\backslash \operatorname{supp} y_\epsilon^*$ (if supp $y_\epsilon^*$ is a finite union of perfect nowhere dense sets) and modify $f$ on $E \cap [t, t + \epsilon]$ and $E \cap [t - \epsilon, t]$, $E$ being some appropriate closed subset of $D$ as above. If supp $y_\epsilon^*$ is not a finite union, use the same approach of considering only finitely many of the infinite terms of $y_\epsilon^*$ that was used above in Case I.

Thus suppose $t \notin \operatorname{supp} z^{-*}$. But then $t \in \operatorname{supp} x^{+*}$ implies $t \in \operatorname{supp} y^{-*}$ and for epsilon sufficiently small $x_\epsilon^{-*} = y_\epsilon^{-*}$. Construct $G$, $H$ by using the $D$ and $\{D_\xi\}_{\xi>0}$ approach as in Case I.

*Case* III. $t \notin \operatorname{supp}(w^* + z^*)$. Since $t \in A \cap B$, $t \in \operatorname{supp} y^{+*} \cap \operatorname{supp} y^{-*}$. Let $D = [a, b]\backslash \operatorname{supp} y^{+*}$, $E = [a, b] \operatorname{supp} y^{-*}$. Since $\| y^* \| = \| y^{+*} \| + \| y^{-*} \|$, $D$ contains all of supp $y^{-*}$ except for a set of measure zero (similarly for $E$ and supp $y^{+*}$). Letting $D'$, $E'$ be closed compact subsets of supp $y^{-*}$, supp $y^{+*}$, contained in $D$ and $E$, and of positive measure, we can find disjoint open neighborhoods $E''$, $E''$ of $D'$, $E'$ and construct our functions $G$, $H$ by modifying $f$ on $D'$, $E'$, respectively, ∎

COROLLARY 2.1. *If* $x^*$ *is a* RRAS *functional*, $f \in C[a, b]$ *and* $\ell$, $u$ *permis-*

*sible bounding functions such that* $\ell \leqslant f \leqslant u$ *on* $[a, b]$, *then there exists a* $\nu > 0$ *such that given* $|\eta| < \nu$ *there exists a polynomial* $p_\eta$ *for which* $\ell \leqslant p_\eta \leqslant u$ *and* $x^*p_\eta = x^*f + \eta$.

## 3. RRAS Sequences

DEFINITION 3.1. A sequence of bounded linear functionals $x_1^*, \ldots, x_n^*$ is said to be a *RRAS sequence* in case any nonzero $x^* \in \langle x_1^*, \ldots, x_n^* \rangle$ is a RRAS functional.

Below we will show that one may approximate any $f \in C[a, b]$ arbitrarily closely in the RRAS scheme. Considering this eventuality, we first look at some properties of RRAS sequences.

PROPOSITION 3.1. *Suppose* $x_1^*, \ldots, x_n^*$ *is a* RRAS *sequence on* $C[a, b]$. *Let* $S = \{s_1, \ldots, s_m\}$ *be a finite subset of* $[a, b]$. *Set* $v_i^* = x_i^* \circ \chi_D$, $D = [a, b] \backslash S$. *Then* $v_1^*, \ldots, v_n^*$ *is also a* RRAS *sequence on* $C[a, b]$.

*Remark* 3.1. If $S_\delta = N_\delta(S)$, $D_\delta = [a, b] \backslash S_\delta$, and $v_{i,\delta}^* = x_i^* \circ \chi_{D_\delta}$, it is not the case that $x_1^*, \ldots, x_n^*$ a RRAS sequence on $C[a, b]$ and $S$ a finite subset of $[a, b]$ implies there is a $\delta > 0$ sufficiently small in order that $v_{1,\delta}^*, \ldots, v_{n,\delta}^*$ is necessarily a RRAS sequence. As a counterexample consider the following;

EXAMPLE 3.1. $n = 1$, $x_1^* = x_n^* = x^* = \int_0^1 \cdot\, dx - \sum_{j=1}^{\infty} 2^{-j} e_{2^{-j}}$. $x^*$ is a RRAS functional, $v^* = x^* \circ \chi_{(0,1]}$ is a RRAS functional, but $v^* = \int_\delta^1 \cdot fx - \sum_{j=1}^{[-\ln\delta]} 2^{-j} e_{2^{-j}}$ has supp $v_\delta^{+*} \cap$ supp $v_\delta^{-*} = \{2^{-1}, \ldots, 2^{-[-\ln \delta]}\}$, a finite point set only, and so by Theorem 2.1 $v_\delta^*$ is not a RRAS functional, for any $\delta > 0$.

PROPOSITION 3.2. *Suppose* $x_1^*, \ldots, x_n^*$ *is a linearly independent* RRAS *sequence on* $C[a, b]$. *Let* $S = \{s_1, \ldots, s_m\}$ *be a finite subset of* $[a, b]$. *Set* $v_i^* = x_i^* \circ \chi_D$, $D = [a, b] \backslash S$. *Then* $v_1^*, \ldots, v_n^*$ *is also a linearly independent* RRAS *sequence on* $C[a, b]$.

COROLLARY 3.1. *If* $S_\delta = N_\delta(S)$, $D_\delta = [a, b] \backslash S_\delta$, *and* $v_{i,\delta}^* = x_i^* \circ \chi_{D_\delta}$, *then* $x_1^*, \ldots, x_n^*$ *a linearly independent* RRAS *sequence on* $C[a, b]$ *and* $S$ *a finite subset of* $[a, b]$ *implies there exists a* $\delta_0 > 0$ *such that* $v_{1,\delta}^*, \ldots, v_{n,\delta}^*$ *is a linearly independent span indefinite sequence whenever* $0 \leqslant \delta \leqslant \delta_0$.

*Proof.* If $v_{1,\delta}^*, \ldots, v_{n,\delta}^*$ is not span indefinite for any $\delta > 0$, let $v_\delta^* = \sum_{i=1}^{n} \alpha_{i,\delta} v_{i,\delta}^*$ be a nonzero positive linear functional on $C[a, b]$. If $\delta' < \delta''$, $v_{\delta''}^* = v_{\delta'}^* \circ \chi_{D_{\delta''}} = \sum_{i=1}^{n} \alpha_{i,\delta''} v_{i,\delta''}^*$ must also be a positive linear functional. Since supp $x^{+*} \cap$ supp $x^{-*}$ is infinite for any nonzero $x^* \in \langle x_1^*, \ldots, x_n^* \rangle$, for each such positive linear functional $v_\delta^*$ there must be a $\delta' < \delta$ for which

$v_\delta^*$ is not a positive linear functional. Thus given $\delta < 0$ arbitrarily small, there exist infinitely many $\{\alpha_{i,\nu}\}_{\nu>0}$ such that $v_\nu^* = \sum_{i=1}^n \alpha_{i,\nu} v_{i,\delta}^*$ are positive linear functionals on $C[a, b]$, and these $v_\nu^*$ generate a nonzero subspace $V_\delta$ contained in $V_{\delta'}$ whenever $\delta < \delta'$. But $\dim \langle v_1^*,..., v_n^*\rangle = n < \infty$, so $\bigcap_{\delta>0} V_\delta$ is also a nonzero subspace $V$ of $\langle v_1^*,..., v_n^*\rangle$. But then some basis of $V$ must consist entirely of positive linear functionals, and so $v_1^*,..., v_n^*$ cannot be a linearly independent RRAS sequence on $C[a, b]$. ∎

*Remark* 3.2. If one finds it difficult to see why the $V$ above must have a basis consisting of positive linear functionals, replace the $V_\delta$ of the above proof by positive cones $W_\delta$ consisting entirely of positive linear functionals. As above $W_\delta \supseteq W_{\delta'}$ whenever $\delta' < \delta$ and no $W_\delta$ is the zero cone (recall that if $u^*$, $v^*$ are linearly independent positive linear functionals, and if there exist countably many distinct positive linear functionals in the positive cone spanned by $u^*$ and $v^*$ which do not all lie in finitely many one-dimensional subspaces of $\langle u^*, v^*\rangle$, then the positive cone spanned by $u^*$, $v^*$ consists entirely of positive linear functionals).

PROPOSITION 3.3. *Suppose $x^*$ is a* RRAS *functional on $C[a, b]$. Let $v^* = x^* \circ \chi_B$, $B$ an open subset of $[a, b]$. Suppose $v^*$ is also a* RRAS *functional on $C[a, b]$. Then there exists a closed subset $E$ of $B$ such that $u^* = v^* \circ \chi_E = x^* \circ \chi_E$ is a (span) indefinite linear functional on $C[a, b]$.*

PROPOSITION 3.4. *Suppose $x_1^*,..., x_n^*$ is a linearly independent* RRAS *sequence on $C[a, b]$. Let $v_i^* = x_i^* \circ \chi_B$, $B$ an open subset of $[a, b]$. Suppose $v_1^*,..., v_n^*$ are also a* RRAS *sequence on $C[a, b]$. Then there exists a closed subset $E$ of $B$ such that $u_1^*,..., u_n^*$ are span indefinite on $C[a, b]$, where $u_i^* = v_i^* \circ \chi_E = x_i^* \circ \chi_E$.*

*Proof.* By Proposition 3.3 there is a closed subset $E'$ of $B$ for which $v_n^* \circ \chi_{E'}$ is an indefinite linear functional. By induction there is a closed subset $E''$ of $B$ for which $v_1^* \circ \chi_{E''},..., v_{i-1}^* \circ \chi_{E''}, v_{i+1}^* \circ \chi_{E''},..., v_n^* \circ \chi_{E''}$ are span indefinite $(i = 0,..., n)$. Let $E''' = E' \cup E''$ and set $u_i^* = v_i^* \circ \chi_{E'''}$. If $u_1^*,..., u_n^*$ is not span indefinite on $C[a, b]$, then some $(\sum_{i=1}^{n-1} \beta_i u_i^*) + u_n^*$ is a (without loss of generality) positive linear functional on $C[a, b]$. Since $\operatorname{supp} u_n^{-*} \cap \operatorname{supp} u_n^{+*}$ is nonempty, necessarily

(i) $(\sum_{i=1}^{n-1} \beta_i u_i^*)^+ = u_n^{-*} + a^*$ for some positive linear functional $a^*$ on $C[a, b]$, and

(ii) $u^{+*} = (\sum_{i=1}^{n-1} \beta_i u_i^*)^- + b^*$ for some positive linear functional $b^*$ on $C[a, b]$.

Since $u_1^*,..., u_{n-2}^*$, $u_n^*$ are span indefinite, fixing $\beta_1,..., \beta_{n-2}$ we find at most two values of $\beta_{n-1}$ can be such that $\sum_{i=1}^{n-1} \beta_i u_i^* + u_n^*$ is not indefinite.

Fixing $\beta_1, \ldots, \beta_{n-3}, \beta_{n-1}$ we likewise find at most two values of $\beta_{n-2}$. For each of those values of $\beta_{n-2}$, fixing $\beta_1, \ldots, \beta_{n-3}$ as before we find at most two values of $\beta_{n-1}$ (for a total of four). In this way at most finitely many $u^* \in \langle u_1^*, \ldots, u_{n-1}^* \rangle$ are such that $u^* + u_n^*$ can fail to be indefinite. For each of them we can choose closed subsets $E_j$ of $B$ for which $(u^* + u_n^*) \circ \chi_{E_j}$ is indefinite, and thus overall setting $E = E''' \cup (\cup_j B_j)$ we find that the induced $u_i^* = v_i^* \circ \chi_E$ are span indefinite on $C[a, b]$. ∎

*Remark* 3.3. Perhaps a more intuitive proof to Proposition 3.4 above is to simply pick a closed subset $E$ of $B$ containing enough atoms of each $v_i^*$ in its interior to render the induced linear functionals $u_i^*$ linearly independent. Since without loss of generality each $v_i^*$ will have atoms the others lack, choosing $E$ to contain the proper atoms in its interior will not only render the induced $u_i^*$ linearly independent but also span indefinite, for each $u_i^*$ will contain atoms lying in supp $u_i^{+*} \cap$ supp $u_i^{-*}$ which will not be atoms of any $u_j^*$ $(j \neq i)$ and hence cannot disappear in any linear combination of the $u_i^*$ without taking a zero coefficient.

We generalize Lemma 2.1 next.

LEMMA 3.1. *If $x_1^*, \ldots, x_n^*$ is a linearly independent* RRAS *sequence on $C[a, b]$, $f \in C[a, b]$ and $\ell, u$ permissible bounding functions such that $\ell \leqslant f \leqslant u$ $[a, b]$, then there exists a $\nu > 0$ such that given $\sigma = (\sigma_1, \ldots, \sigma_n) \in \{-1, 1\}^n$ there exists a continuous function j for which both*

(i) *$\ell \leqslant h_\sigma \leqslant u$ on $[a, b]$, and* (ii) *$\sigma_j x_j^*(h_\sigma - f) > \nu$ $(j = 1, \ldots, n)$.*

*Proof.* Set $A = \{x \in [a, b]: f(x) = \ell(x)\}$, $B = \{x \in [a, b]: \ell(x) < f(x) < u(x)\}$, $C = \{x \in [a, b]: f(x) = u(x)\}$; $D = A \cup C$, $T = \{x \in [a, b]: \ell(x) = u(x)\} = A \cap C$, $r_i^* = x_i^* \circ \chi_B$. Suppose $r_1^*, \ldots, r_\mu^*$ is a maximal linearly independent RRAS sequence among the $r_1^*, \ldots, r_n^*$, $0 \leqslant \mu \leqslant n$. By Proposition 3.4 let $E$ be a closed subset of $B$ for which $r_1^*, \ldots, r_\mu^*$ is a (linearly independent) span indefinite sequence on $C[a, b]$. By Corollary 3.1 let $\delta > 0$ be such that $x_1^* \circ \chi_H, \ldots, x_n^* \circ \chi_H$ is a (linearly independent) span indefinite sequence of linear functionals on $C[a, b]$, where $H = [a, b] \backslash N_\delta(T)$. Since $T$ and $E$ are disjoint closed subsets of $[a, b]$, suppose that $\delta > 0$ is sufficiently small such that $H$ contains $E$ in its interior (normality of the interval $[a, b]$). Set $s_i^* = x_i^* \circ \chi_I$, $I = (H \cap D) \cup E$, and $\psi = (\frac{1}{2}) \min \{\min\{(u - \ell)(x): x \in H \cap D\}, \min\{(u - f)(x): x \in E\}, \min\{(f - \ell)(x): x \in E\}\}$. By separately analyzing the $x_1^*, \ldots, x_\mu^*$ and the $x_{\mu+1}^*, \ldots, x_n^*$ we observe that the $s_1^*, \ldots, s_n^*$ may be assumed to be a (linearly independent) span indefinite sequence on $C[a, b]$. Writing each functional as $s_i^* = s_i^* \circ \chi_{A \cap H} + s_i^* \circ \chi_{C \cap H} + s_i^* \circ \chi_E$, defining $t_i^* = s_i^* \circ \chi_{A \cap H} - s_i^* \circ \chi_{C \cap H} + s_i^* \circ \chi_E$ we have that $t_i^*, \ldots, t_n^*$ is also a (linearly independent) span indefinite sequence on $C[a, b]$.

By linear independence choose $p_{i,\iota} \in C[a, b]$ $(i = 1,..., n;\ \iota = -1, 1)$ so that $s_j{}^*p_{i,\iota} = \iota\delta_{ij}$. By span indefiniteness (Proposition 1.2) choose $k' \in C[a, b]$ so that (i) $k' \geqslant 1$ on $[a, b]$, and (ii) $t_j{}^*k' = 0$ $(j = 1,..., n)$. Since $A \cap H$, $C \cap H$, $E$ are mutually disjoint compact subsets of $[a, b]$ whose union contains the support of all the $s_k{}^*$ and $t_j{}^*$, choose a $k \in C[a, b]$ so that (i) $k(x) = -k'(x)$ if $x \in C \cap H$, (ii) $k(x) = k'(x)$ if $x \in (A \cap H) \cup E$, and (iii) $\| k \| = \| k' \|$. Since $s_j{}^*k = t_j{}^*k'$ and $\psi$ is positive, setting $q_{i,\iota} = \alpha(p_{i,\iota} + \beta k)$ we may choose positive constants $\alpha$ and $\beta$ so that (i) $0 < q_{i,\iota}(x) \leqslant \psi$ for $x \in A \cap H$, (ii) $-\psi \leqslant q_{i,\iota}(x) < 0$ for $x \in C \cap H$, (iii) $-\psi \leqslant q_{i,\iota}(x) \leqslant \psi$ for $x \in E$, (iv) $s_j{}^*p_{i,\iota} = 0$ if $j \neq i$, and (v) $\iota s_i{}^*p_{i,\iota} > 0$. Set $\nu = (2n)^{-1} \min \{| x_j{}^*(q_{i,\iota})| : i = 1,..., n$ and $\iota = -1, 1\}$. Consider the permissible bounding functions $U(x) = u(x) - f(x)$, $L(x) = \ell(x) - f(x)$. Observe that (i) $L$, $U$ has the same nodes as $\ell$, $u$ (the set $T$), (ii) $L(x) = 0$ and $2\psi \leqslant U(x)$ for $x \in A \cap H$, (iii) $L(x) \leqslant -2\psi$ and $U(x) = 0$ for $x \in C \cap H$, and (iv) $L(x) \leqslant -2\psi < 0 < 2\psi \leqslant U(x)$ for $x \in E$. Since $L(x) < q_{i,\iota}(x) < U(x)$ for $x \in I$ a compact subset of $[a, b]$, and $L(x) \leqslant 0 \leqslant U(x)$ globally on $[a, b]$, for $\eta > 0$ sufficiently small we can find continuous functions $h_{i,\iota,\eta}$ so that (i) $h_{i,\iota,\eta}(x) = q_{i,\iota}(x)$ if $x \in I$, (ii) $h_{i,\iota,\eta}(x) = 0$ if $\operatorname{dist}(x, I) \geqslant \eta$, and (iii) $L(x) \leqslant h_{i,\iota,\eta}(x) \leqslant U(x)$ otherwise. Since $J_\eta = \{x \in [a, b]\colon \operatorname{dist}(x, I) < \eta\}$ is a decreasing sequence of open subsets of $[a, b]$ whose limit (intersection) is $I$, $x_j{}^*h_{i,\iota,\eta} \to s_j{}^*h_{i,\iota,\eta}$ as $\eta \to 0$. Fix $\eta > 0$ so that $|(x_j{}^* - s_j{}^*)\, h_{i,\iota,\eta}| < n^{-1} \cdot 10^{-6}\nu$ uniformly in $i$ and $\iota$ and set $h_\sigma = n^{-1} \sum_{i=1}^{n} h_{i,\sigma_i,\eta}$. Then $L \leqslant h_\sigma \leqslant U$ and $x_j{}^*h_\sigma = n^{-1}(x_j{}^*h_{i,\sigma_j,\eta}) + n^{-1} \sum_{i=1, i\neq j}^{n} x_j{}^*h_{i,\sigma_i,\eta}$. Since the last term has magnitude at most $10^{-6}\nu$ while the first term has magnitude at least $2\nu$, with sign $\sigma_j$, we find $\sigma_j x_j{}^*h_\sigma > \nu$ $(j = 1,..., n)$ and the conclusion of the lemma follows. ∎

THEOREM 3.1. *Suppose $x_1{}^*,..., x_n{}^*$ is a* RRAS *sequence of linear functionals on $C[a, b]$. Then given $f \in C[a, b]$ and permissible $\ell$, $u$ for which $\ell \leqslant f \leqslant u$ there necessarily exists a polynomial $p \in \Pi$ for which $\ell \leqslant p \leqslant u$ and $x_i{}^*p = x_i{}^*f$ $(i = 1,..., n)$.*

*Proof.* Without loss of generality suppose the $x_1{}^*,..., x_n{}^*$ are linearly independent on $C[a, b]$. Let $\sigma = (\sigma_1,..., \sigma_n) \in \{-1, 1\}^n$ be arbitrary. Set $\tau = (\sigma_1,..., \sigma_{n-1}, -\sigma_n)$ and choose continuous functions $h_\sigma$, $h_\tau$ by Lemma 3.1. By Proposition 1.2, we may find polynomials $p_\sigma$, $p_\tau$ for which $\ell \leqslant p_\sigma$, $p_\tau \leqslant u$ and $\sigma_j x_j{}^*(p_\sigma - f) > \nu$, $\tau_j x_j{}^*p_\tau > \nu$ $(j = 1,..., n)$. Let $0 < \lambda < 1$ be such that $x_n{}^*(\lambda p_\sigma + (1 - \lambda)\, p_\tau - f) = 0$ and set $p_\sigma' = \lambda p_\sigma + (1 - \lambda)\, p_\tau$. Then (i) $\ell \leqslant p_\sigma' \leqslant u$, (ii) $x_n{}^*(p_\sigma' - f) = 0$, (iii) $\sigma_j x_j{}^*(p_\sigma' - f) > \nu$ $(j = 1, ..., n - 1)$, and (iv) $(\sigma_1,..., \sigma_{n-1}) \in \{-1, 1\}^{n-1}$ is arbitrary. By induction there is a polynomial $p \in \Pi$ for which $\ell \leqslant p \leqslant u$ and $x_j{}^*(p - f) = 0$ $(j = 1, ..., n)$. ∎

COROLLARY 3.2. *Suppose $x_1{}^*,..., x_n{}^*$ is a* RRAS *sequence of linear* *functionals on* $C[a, b]$. *Then given* $f \in C[a, b]$, *permissible* $\ell$, $u$ *for which* $\ell \leqslant f \leqslant u$, *and* $\epsilon > 0$ *arbitrary there necessarily exists a polynomial* $p \in \Pi$ *for which* (i) $\ell \leqslant p \leqslant u$, (ii) $x_j{}^*p = x_j{}^*f$ $(j = 1,..., n)$, *and* (iii) $\| f - p \| \leqslant \epsilon$.

*Remark* 3.4. Corollary 3.2 is our desired Weierstrass theorem for RRAS approximation with arbitrary permissible bounding functions. Notice the manner we have derived our Weierstrass theorem as a corollary of Theorem 3.1 parallels the derivation of the Weierstrass-type theorem Proposition 1.2 in [1] as a corollary to a theorem (Proposition 1.1) similar in statement to Theorem 3.1 above. Such an approach (obtaining Weierstrass-type theorems as corollaries of theorems analogous to Theorem 3.1 above) is clearly useable for any approximation process whose side conditions are amenable to ("invariant" under) convex linear combinations.

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